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A new formulation of the path integral method for the chiral anomaly

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Abstract. Using a comoving representation we reformulate the path integral method for the chiral anomaly and show that the anomaly comes from the accumulating effect in argument change of diagonalised matrix elements for non-Hermitian operators. The analysis of the Wess-Zumino consistency condition reveals further that it is the contribution of zero modes which needs to be considered. Thus a new regularisation scheme is proposed which leads to a unified expression for the anomaly in $2n$ dimensions both for Abelian and non-Abelian cases.

1. Introduction

In recent years investigations of the chiral anomaly has become one of the most important problems in particle physics (as well as in other branches of theoretical physics). There are several kinds of approach to attack this problem. The path integral formulation initiated by Fujikawa (1979, 1980a, b) is highly appreciated due to its elegance and non-perturbative nature. However, some authors have found their results depend on the functional measure and regularisation scheme they used (Gomboa Saravi *et al* 1984, Mckay and Young 1983, Fujikawa 1984). So the Wess-Zumino condition has received much attention (Balachandran *et al* 1982, Andrianov *et al* 1982, Andrianov and Bonora 1984, Hu *et al* 1984, Eindorn and Jones 1984). In this paper we will propose a new scheme of the path-integral method aiming at getting rid of some ambiguities and deriving a unified formula for manipulating both the Abelian and non-Abelian chiral anomaly in $2n$ dimensions.

The organisation of this paper is as follows. In § 2 a comoving representation instead of a functional measure change is explained in some detail before we can prove in § 3 that the unique role played by the zero modes of the Dirac operator is either Hermitian or not. It is the argument changes of these zero modes which contribute to the chiral anomaly. After a short excursion to the special case of Abelian anomaly in § 4, in § 5 we are led to propose a new 'regularisation' scheme which picks only these contributions. Actually, it is finite and no divergence problem is involved. Section 6 contains a summary and discussion. Some mathematical details are given in the appendices.

2. The comoving representation

Consider the action for an N-component Dirac field with both vector and axial vector

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coupling in $2n$-dimensional Euclidean space:

$$S(\psi, \bar{\psi}, V, A) = \int \mathrm{d}^{2n}x\, \bar{\psi}\mathrm{i}\not{D}\psi \tag{2.1}$$

where

$$\mathrm{i}\not{D} = \mathrm{i}\not{\partial} + \mathrm{i}\not{V} + \mathrm{i}\not{A}\gamma_5 \tag{2.2}$$

with

$$\gamma_\mu = \gamma_\mu^\dagger \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \qquad \gamma_5 = \mathrm{i}^n \prod_{\mu=1}^{2n} \gamma_\mu = \gamma_5^\dagger$$

and $V_\mu = V_\mu^a \lambda^a$, $A_\mu = A_\mu^a \lambda^a$ in which the λ^a are anti-Hermitian generators of the gauge group SU(N). Both $V_\mu^{a(x)}$ and $A_\mu^{a(x)}$ are real external gauge potentials. The vector and axial vector currents composed of bilinear fermion fields are defined as follows:

$$j_\mu^a = \bar{\psi}\lambda^a\gamma_\mu\psi \qquad j_\mu^{5a} = \bar{\psi}\lambda^a\gamma_\mu\gamma_5\psi. \tag{2.3}$$

On the classical level, under the infinitesimal field transformation

$$\psi \to \exp(\alpha(x))\psi \qquad \bar{\psi} \to \bar{\psi}\exp(-\alpha(x)) \tag{2.4}$$

with $\alpha(x) = \alpha^{a(x)}\lambda^a$, the invariance condition of action leads to

$$\mathrm{D}_\mu j_\mu = \partial_\mu j_\mu + [j_\mu, V_\mu] + [j_\mu^5, A_\mu] = 0. \tag{2.5}$$

Similarly the invariance of S under the infinitesimal chiral transformation

$$\psi \to \exp(\beta(x)\gamma_5)\,\psi \qquad \bar{\psi} \to \bar{\psi}\exp(\beta(x)\gamma_5) \tag{2.6}$$

with $\beta(x) = \beta^{a(x)}\lambda^a$ leads to

$$\mathrm{D}_\mu j_\mu^5 = \partial_\mu j_\mu^5 + [j_\mu^5, V_\mu] + [j_\mu, A_\mu] = 0. \tag{2.7}$$

However, on the quantum level, we have to deal with the following generating functional:

$$W(V, A) = \exp(-\Gamma(V, A)) = \int [\mathrm{d}\bar{\psi}\,\mathrm{d}\psi] \exp\left(-\int \mathrm{d}^{2n}x\, \bar{\psi}\mathrm{i}\not{D}\psi\right). \tag{2.8}$$

Instead of looking for the Jacobian accompanying the change of integral measure under infinitesimal field transformation, we propose a comoving representation approach. After one step of infinitesimal transformation of either (2.4) or (2.6), one obtains from (2.8)

$$W(V, A) = \frac{W(V, A)}{W(V, A; a_1)}\, W(V, A; a_1)$$

with

$$W(V, A; a_1) = \int [\mathrm{d}\bar{\psi}\,\mathrm{d}\psi] \exp\left(-\int \mathrm{d}^{2n}x\, \bar{\psi}_i\not{D}(a_1)\psi\right) \tag{2.9}$$

where

$$\mathrm{i}\not{D}(\alpha_1) = \exp(\alpha_1(x))\,\mathrm{i}\not{D}\exp(-\alpha_1(x)) \tag{2.10}$$

or

$$\mathrm{i}\not{D}(\beta_1) = \exp(-\beta_1(x)\gamma_5)\,\mathrm{i}\not{D}\exp(-\beta_1(x)\gamma_5). \tag{2.11}$$

More generally, after s steps, one can write

$$W(V, A) = \exp\left(\sum_{r=1}^{s} \delta\Gamma(V, A; a_1, \ldots, a_\gamma)\right) W(V, A; a_1, \ldots, a_s) \tag{2.12}$$

where

$$\exp(\delta\Gamma(V, A; a_1, \ldots, a_\gamma)) = \frac{W(V, A; a_1, \ldots, a_{\gamma-1})}{W(V, A; a_1, \ldots, a_\gamma)} \tag{2.13}$$

with

$$W(V, A; a_1, \ldots, a_\gamma) = \int [\mathrm{d}\bar{\psi}\, \mathrm{d}\psi] \exp\left(-\int \mathrm{d}^{2n}x\, \bar{\psi} \mathrm{i}\not{D}(a_1, \ldots, a_\gamma)\psi\right) \tag{2.14}$$

and

$$\mathrm{i}\not{D}(a_1, \ldots, a_{\gamma-1}; \alpha_\gamma) = \exp(\alpha_\gamma(x))\mathrm{i}\not{D}(a_1, \ldots, a_{\gamma-1}) \exp(-\alpha_\gamma(x)) \tag{2.15}$$

or

$$\mathrm{i}\not{D}(a_1, \ldots, a_{\gamma-1}; \beta_\gamma) = \exp(-\beta_\gamma(x)\gamma_5)\mathrm{i}\not{D}(a_1, \ldots, a_{\gamma-1}) \exp(-\beta_\gamma(x)\gamma_5) \tag{2.16}$$

with a_i being $\alpha_i(x)$ or $\beta_i(x)$ $(i = 1, 2, \ldots, s)$ respectively.

Notice that $\mathrm{i}\not{D}(a_1, \ldots, a_\gamma)$ is a non-Hermitian Dirac operator in general. So in order to calculate the ratio (2.13), we introduce two Hermitian operators $\Delta(a_1, \ldots, a_\gamma)$ and $\tilde{\Delta}(a_1, \ldots, a_\gamma)$ as follows:

$$\Delta(a_1, \ldots, a_\gamma) = (\mathrm{i}\not{D})^\dagger(a_1, \ldots, a_\gamma)\mathrm{i}\not{D}(a_1, \ldots, a_\gamma) \tag{2.17}$$

$$\tilde{\Delta}(a_1, \ldots, a_\gamma) = \mathrm{i}\not{D}(a_1, \ldots, a_\gamma)(\mathrm{i}\not{D})^\dagger(a_1, \ldots, a_\gamma) \tag{2.18}$$

with the eigenequations:

$$\Delta(a_1, \ldots, a_\gamma)\phi_n(x; a_1, \ldots, a_\gamma) = \lambda_n^2 \phi_n(x; a_1, \ldots, a_\gamma) \tag{2.19}$$

$$\tilde{\Delta}(a_1, \ldots, a_\gamma)\tilde{\phi}_n(x; a_1, \ldots, a_\gamma) = \lambda_n^2 \tilde{\phi}_n(x; a_1, \ldots, a_\gamma). \tag{2.20}$$

It is easy to check the mapping relations

$$\mathrm{i}\not{D}(a_1, \ldots, a_\gamma)\phi_n(x; a_1, \ldots, a_\gamma) = \lambda_n \exp[\mathrm{i}\theta_n(a_1, \ldots, a_\gamma)]\tilde{\phi}_n(x; a_1, \ldots, a_\gamma) \tag{2.21}$$

$$(\mathrm{i}\not{D})^\dagger(a_1, \ldots, a_\gamma)\tilde{\phi}_n(x; a_1, \ldots, a_\gamma) = \lambda_n \exp[-\mathrm{i}\theta_n(a_1, \ldots, a_\gamma)]\phi_n(x; a_1, \ldots, a_\gamma) \tag{2.22}$$

and the recursion relations

$$\phi_n(x; a_1, \ldots, a_{\gamma-1}, \alpha_\gamma) = \exp(\tfrac{1}{2}\mathrm{i}\delta\theta_n(a_1, \ldots, a_{\gamma-1}, \alpha_\gamma)) \times \exp(\alpha_\gamma(x))\phi_n(x, a_1, \ldots, a_{\gamma-1}) \tag{2.23}$$

$$\tilde{\phi}_n(x; a_1, \ldots, a_{\gamma-1}, \alpha_\gamma) = \exp(-\tfrac{1}{2}\mathrm{i}\delta\theta_n(a_1, \ldots, a_{\gamma-1}, \alpha_\gamma)) \times \exp(\alpha_\gamma(x))\, \tilde{\phi}_n(x; a_1, \ldots, a_{\gamma-1}) \tag{2.24}$$

$$\phi_n(x; a_1, \ldots, a_{\gamma-1}, \beta_\gamma) = \exp(\tfrac{1}{2}\mathrm{i}\delta\theta_n(a_1, \ldots, a_{\gamma-1}, \beta_\gamma)) \times \exp(\beta_\gamma(x)\gamma_5)\phi_n(x; a_1, \ldots, a_{\gamma-1}) \tag{2.25}$$

$$\tilde{\phi}_n(x; a_1, \ldots, a_{\gamma-1}, \beta_\gamma) = \exp(-\tfrac{1}{2}\delta\theta_n(a_1, \ldots, a_{\gamma-1}, \beta_\gamma)) \times \exp(-\beta_\gamma(x)\gamma_5)\, \tilde{\phi}_n(x; a_1, \ldots, a_{\gamma-1}) \tag{2.26}$$

with

$$\delta\theta_n(a_1, \ldots, a_{\gamma-1}, a_\gamma) = \theta_n(a_1, \ldots, a_\gamma) - \theta_n(a_1, \ldots, a_{\gamma-1}) \tag{2.27}$$

where a_i are either $\alpha_i(x)$ or $\beta_i(x)$ ($i = 1, 2, \ldots, \gamma$). We wish to add that while the relations (2.17)-(2.22) were familiar in the literature, the arguments θ_n were overlooked. It is the change in arguments θ_n and the accumulating effect of them which occupy the central position in the following discussion. Actually, noting that

$$\int \mathrm{d}^{2n}x\tilde{\phi}_m^{\dagger}(x; a_1, \ldots, a_\gamma)\mathrm{i}\not{D}(a_1, \ldots, a_\gamma)\phi_n(x; a_1, \ldots, a_\gamma) = \lambda_n \exp[\mathrm{i}\theta_n(a_1, \ldots, a_\gamma)]\delta_{mn} \tag{2.28}$$

and expanding

$$\psi(x) = \sum_n a_n\phi_n(x; a_1, \ldots, a_\gamma) \qquad \bar{\psi}(x) = \sum_n \tilde{\phi}_n^{\dagger}(x; a_1, \ldots, a_\gamma)\bar{b}_n \tag{2.29}$$

with a_n and $\bar{b}_n$ being anticommuting variables, we can formally write

$$W(V, A; a_1, \ldots, a_s) = \det \mathrm{i}\not{D}(a_1, \ldots, a_s) = \prod_n \lambda_n \exp[\mathrm{i}\theta_n(a_1, \ldots, a_s)] \tag{2.30}$$

and express the ratio (2.13) as

$$\exp(\delta\Gamma(V, A; a_1, \ldots, a_s)) = \prod_n \frac{\lambda_n \exp[\mathrm{i}\theta_n(a_1, \ldots, a_{s-1})]}{\lambda_n \exp[\mathrm{i}\theta_n(a_1, \ldots, a_{s-1}, a_s)]} = \prod_n \exp[-\mathrm{i}\delta\theta_n(a_1, \ldots, a_{s-1}, a_s)]. \tag{2.31}$$

Denoting

$$\delta\mathrm{i}\not{D}(a_1, \ldots, a_{s-1}, a_s) = \mathrm{i}\not{D}(a_1, \ldots, a_{s-1}, a_s) - \mathrm{i}\not{D}(a_1, \ldots, a_{s-1}) \tag{2.32}$$

and treating it as a perturbation, one finds

$$\lambda_n \exp[\mathrm{i}\theta_n(a_1, \ldots, a_{s-1})]\mathrm{i}\delta\theta_n(a_1, \ldots, a_s) = \int \mathrm{d}^{2n}x\, \tilde{\phi}_n^{\dagger}(x; a_1, \ldots, a_{s-1})\delta\mathrm{i}\not{D}(a_1, \ldots, a_s)\phi_n(x; a_1, \ldots, a_{s-1}). \tag{2.33}$$

For the case of $a_s = \alpha_s(x)$ in (2.15)

$$\delta\mathrm{i}\not{D}(a_1, \ldots, a_{s-1}, \alpha_s) = [\mathrm{i}\not{D}(a_1, \ldots, a_{s-1}), -\alpha_s] \tag{2.34}$$

so, with the help of (2.21) and (2.22) the $\mathrm{i}\delta\theta_n$ in (2.33) is

$$\mathrm{i}\delta\theta_n(a_1, \ldots, a_{s-1}, \alpha_s) = -\int \mathrm{d}^{2n}x\, \phi_n^{\dagger}(x; a_1, \ldots, a_{s-1})\alpha_s(x)\phi_n(x; a_1, \ldots, a_{s-1}) + \int \mathrm{d}^{2n}x\, \tilde{\phi}_n^{\dagger}(x; a_1, \ldots, a_{s-1})\alpha_s(x)\tilde{\phi}_n(x; a_1, \ldots, a_{s-1}). \tag{2.35}$$

For the case of $a_s = \beta_s(x)$ in (2.16)

$$\delta\mathrm{i}\not{D}(a_1, \ldots, a_{\gamma-1}, \beta_s) = -\{\mathrm{i}\not{D}(a_1, \ldots, a_{s-1}), \beta_s\gamma_5\} \tag{2.36}$$

so

$$\mathrm{i}\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_5) = -\int \mathrm{d}^{2n}x\, \phi_n^{\dagger}(x; a_1, \ldots, a_{s-1})\beta_s(x)\gamma_5\phi_n(x; a_1, \ldots, a_{s-1}) - \int \mathrm{d}^{2n}x\, \tilde{\phi}_n^{\dagger}(x; a_1, \ldots, a_{s-1})\beta_s(x)\gamma_5\tilde{\phi}_n(x; a_1, \ldots, a_{s-1}). \tag{2.37}$$

Some ambiguities may arise from the ratio of zero diagonal elements in (2.31). However as discussed in appendix 1, (2.37) is valid also for zero modes. In fact, the arguments of zero modes are the dominant contribution to anomaly.

Evidently, for the case of $a_s = \alpha_s(x)$, we find from (2.31) and (2.35)

$$\begin{aligned}\delta\Gamma(V, A; a_1, \ldots, a_{s-1}, \alpha_s)\\ &= \sum_n \int \mathrm{d}^{2n}x\, \phi_n^+(x; a_1, \ldots, a_{s-1})\alpha_s(x)\phi_n(x; a_1, \ldots, a_{s-1})\\ &\quad -\sum_n \int \mathrm{d}^{2n}x\, \tilde{\phi}_n^+(x; a_1, \ldots, a_{s-1})\alpha_s(x)\tilde{\phi}_n(x; a_1, \ldots, a_{s-1})\\ &= \mathrm{Tr}\ \alpha_s - \mathrm{Tr}\ \alpha_s = 0. \end{aligned} \tag{2.38}$$

This means that in the V–A scheme, one can always choose such a representation without the appearance of a vector anomaly. However instead, for the case of $a_s = \beta_s(x)$, we find

$$\begin{aligned}\delta\Gamma(V, A; a_1, \ldots, a_{s-1}, \beta_s)\\ &= \sum_n \int \mathrm{d}^{2n}x\, \phi_n^+(x; a_1, \ldots, a_{s-1})\beta_s(x)\gamma_5\phi_n(x; a_1, \ldots, a_{s-1})\\ &\quad +\sum_n \int \mathrm{d}^{2n}x\, \tilde{\phi}_n^+(x; a_1, \ldots, a_{s-1})\beta_s(x)\gamma_5\tilde{\phi}_n(x; a_1, \ldots, a_{s-1})\\ &= \mathrm{Tr}\ \beta_s\gamma_5 + \mathrm{Tr}\ \beta_s\gamma_5 \end{aligned} \tag{2.39}$$

which does not vanish in general. So the axial vector anomaly appears in comoving representation as a result of argument change of the Dirac operator induced by chiral rotation.

3. The arguments of zero modes and Wess–Zumino condition

For the non-Abelian case, $\beta_s(x)$ in (2.39) is a matrix. Noting that (2.23)-(2.26) imply that eigenfunctions $\{\phi_n(x; a_1, \ldots, a_s)\}$ and $\{\tilde{\phi}_n(x; a_1, \ldots, a_s)\}$ form a basis of projection representation of group SU(N), each one of them acquires a phase under a group transformation, e.g.

$$\begin{aligned}\exp(\beta_s(x)\gamma_5)\phi_n(x; a_1, \ldots, a_{s-1})\\ &= \exp[-\tfrac{1}{2}\mathrm{i}\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s)]\phi_n(x; a_1, \ldots, a_{s-1}, \beta_s). \end{aligned} \tag{3.1}$$

Let us prove that $\Sigma_n\ \delta\theta_n$ is a 1-cocycle over the Lie algebra of the gauge group. Actually, for two infinitesimal transformations β_s and β_{s+1}, we have

$$\begin{aligned}\exp(\beta_{s+1}\gamma_5)\exp(\beta_s\gamma_5) &= \exp(\tfrac{1}{2}[\beta_{s+1}, \beta_s])\exp[(\beta_{s+1}+\beta_s)\gamma_5]\\ &\equiv \exp(\alpha_s)\exp[(\beta_{s+1}+\beta_s)\gamma_5] \end{aligned} \tag{3.2}$$

or

$$\exp[-(\beta_{s+1}+\beta_s)\gamma_5]\exp(\beta_{s+1}\gamma_5)\exp(\beta_s\gamma_5) = \exp(\tfrac{1}{2}[\beta_{s+1}, \beta_s]) \equiv \exp(\alpha_s) \tag{3.3}$$

neglecting higher-order terms of $\mathrm{O}(\beta_i^2)$.

According to (2.31), (2.35) and (2.39), (3.2) corresponds to

$$\sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s) + \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s, \beta_{s+1})]$$
$$= \sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1}) + \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1}, \alpha_s)]. \quad (3.4)$$

If we change the order of β_{s+1} and β_s in (3.2)

$$\exp(\beta_s \gamma_5) \exp(\beta_{s+1} \gamma_5) = \exp(-\alpha_s) \exp[(\beta_s + \beta_{s+1}) \gamma_5] \quad (3.5)$$

we have

$$\sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_{s+1}) + \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_{s+1}, \beta_s)]$$
$$= \sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1})$$
$$- \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1}, \alpha_s)]. \quad (3.6)$$

Equation (3.4) minus (3.6) gives

$$\sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s) + \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s, \beta_{s+1})]$$
$$- \sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_{s+1}) + \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_{s+1}, \beta_s)]$$
$$= 2 \sum_n \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1}, \alpha_s) = 0. \quad (3.7)$$

The last equation comes from (2.38). Equation (3.7) corresponds to

$$\delta\Gamma(V, A; a_1, \ldots, a_{s-1}, \beta_s, \beta_{s+1}) - \delta\Gamma(V, A; a_1, \ldots, a_{s-1}, \beta_{s+1}, \beta_s)$$
$$= 2\delta\Gamma(V, A; a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1}, \alpha_s) = 0 \quad (3.8)$$

which in turn implies the Wess-Zumino consistency condition:

$$[Y(\beta_s), Y(\beta_{s+1})] W(V, A; a_1, \ldots, a_{s+1}) = 0. \quad (3.9)$$

Rewrite (3.4) into a form of coboundary operation:

$$\Delta_{\beta_{s+1}} \delta\Gamma(a_1, \ldots, a_{s-1}, \beta_s) \equiv -\mathrm{i} \sum_n \Delta_{\beta_{s+1}} \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s)$$
$$= -\mathrm{i} \sum_n [\delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s) - \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1})$$
$$+ \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s, \beta_{s+1})]$$
$$= -\mathrm{i} \sum_n \delta\theta_n(a_1, \ldots, a_{s-1}, \beta_s + \beta_{s+1}, \alpha_s) = 0 \quad (3.10)$$

which shows that the Wess-Zumino condition is equivalent to the 1-cocycle of anomalous effective action induced by chiral rotation (Faddeev 1984, Zumino 1985, Jackiw 1986). Now we make a further observation that $\Delta_\beta \delta\theta_n$ may be regarded as a mapping from a closed path composed of three infinitesimal displacements $(\beta_s, \beta_{s+1}, -\beta_s - \beta_{s+1})$ in Lie algebra space into the argument change on the complex plane of $\lambda_n \exp(i\theta n)$. For any $\lambda_n \neq 0$, $\Delta_\beta \delta\theta_n = 0$ is satisfied trivially because the image path does not enclose the origin, whereas for $\lambda_{0i} = 0$ $(i = 0, 1, \ldots, n_+ + n_-)$ the closed path is not topologically trivial; it encloses the origin so $\Delta_\beta \delta\theta_{oi} = \pm 2\pi$, where '±' sign depends on the chirality of the zero mode of $\mathrm{i}\not{D}$ or $(\mathrm{i}\not{D})^\dagger$. Thus the non-trivial constraint imposed by the consistency condition (3.10) is really

$$\Delta_{\beta_{s+1}}\delta\Gamma(a_1,\ldots,a_{s-1},\beta_s)=-\mathrm{i}\sum_{i-1}^{n_++n_-}\Delta_{\beta_{s+1}}\delta\theta_{0i}=0 \qquad \bmod 2\pi\mathrm{i}\times\text{integer} \tag{3.11}$$

Further argument is given in appendix 2. Correspondingly, we assert that for calculating the so-called consistency anomaly it is only meaningful to preserve the contribution of zero modes in (2.39):

$$\delta\Gamma(V,A;a_1,\ldots,a_{s-1},\beta_s)=\operatorname{Tr}\beta_s\gamma_5\vartheta+\operatorname{Tr}\beta_s\gamma_5\tilde{\vartheta} \tag{3.12}$$

where

$$\vartheta\equiv\sum_{i=1}^{n_++n_-}\phi_{0i}(x;a_1,\ldots,a_{s-1})\phi_{0i}^\dagger(x;a_1,\ldots,a_{s-1}) \tag{3.13}$$

or

$$\tilde{\vartheta}\equiv\sum_{j=1}^{\tilde{n}_++\tilde{n}_-}\tilde{\phi}_{0j}(x;a_1,\ldots,a_{s-1})\tilde{\phi}_{0j}^\dagger(x;a_1,\ldots,a_{s-1}) \tag{3.13'}$$

and n_i or $\tilde{n}_i$ are the number of zero modes of $\mathrm{i}\not{D}(a_1,\ldots,a_{s-1})$ or $(\mathrm{i}\not{D})^\dagger(a_1,\ldots,a_{s-1})$ with '±' chirality respectively.

4. The Abelian anomaly

If we discuss the Abelian anomaly case, i.e. $A=0$, $V\neq 0$ in § 2, and $\beta(x)=\mathrm{i}\eta(x)t$ with $\eta(x)$ a real finite c-number function and t being a continuous parameter running from $0\to 1$. Then from (2.31) and (2.39) we define

$$\delta\Gamma(V,\eta(x),t)=-\mathrm{i}\sum_n\delta\theta_n \tag{4.1}$$

where

$$\delta\theta_n=\theta_n(t+\mathrm{d}t)-\theta_n(t)=-\int \mathrm{d}^{2n}x\,\eta(x)[\tilde{\phi}_n^\dagger(x,t)\gamma_5\tilde{\phi}_n(x,t)+\phi_n^\dagger(x,t)\gamma_5\phi_n(x,t)]\,\mathrm{d}t \tag{4.2}$$

with

$$(\mathrm{i}\not{D}')\phi_n(x,t)=\lambda_n\exp(\mathrm{i}\theta_n(t))\,\tilde{\phi}_n(x,t) \tag{4.3}$$

$$(\mathrm{i}\not{D}')^\dagger\tilde{\phi}_n(x,t)=\lambda_n\exp(-\mathrm{i}\theta_n(t))\phi_n(x,t) \tag{4.4}$$

where

$$\mathrm{i}\not{D}'=\exp(-\mathrm{i}\eta(x)t\gamma_5)\mathrm{i}\not{D}\exp(-\mathrm{i}\eta(x)t\gamma_5)=\mathrm{i}\not{D}-\gamma_5t\not{\partial}\eta(x). \tag{4.5}$$

According to the discussion in § 3, we introduce

$$\vartheta(x,t)=\sum_{i=1}^{n_++n_-}\phi_{0i}(x,t)\phi_{0i}^\dagger(x,t) \tag{4.6}$$

or

$$\tilde{\vartheta}(x,t)=\sum_{j=1}^{\tilde{n}_++\tilde{n}_-}\tilde{\phi}_{0j}(x,t)\tilde{\phi}_{0j}^\dagger(x,t) \tag{4.7}$$

as the projection operator of null space of $\mathrm{i}\not{D}'$ or $(\mathrm{i}\not{D}')^{\dagger}$ and $n_{\pm}$ or $\tilde{n}_{\pm}$ being the number of zero modes with $\pm$ chirality respectively. Then one can express (4.1) as

$$\delta\Gamma(V, \eta(x), t) = \mathrm{i}\,\mathrm{d}t \int \mathrm{d}^{2n}x\, \eta(x)(G(x, t) + \tilde{G}(x, t)) \tag{4.8}$$

where

$$G(x, t) = \mathrm{tr}(\gamma_5 \vartheta(x, t)) \tag{4.9}$$

or

$$\tilde{G}(x, t) = \mathrm{tr}(\gamma_5 \tilde{\vartheta}(x, t)) \tag{4.10}$$

is just the so-called local index density of non-Hermitian operator $(\mathrm{i}\not{D}')$ or $(\mathrm{i}\not{D}')^{\dagger}$ respectively.

Actually, in the Abelian case, $A = 0$, $\mathrm{i}\not{D}'^{t=0}$ is Hermitian, so $G(x, t) = \tilde{G}(x, t)$, and it is independent of t. Thus one can perform the trivial integration with respect to t and obtain

$$\Delta\Gamma = \int_{t=0}^{t=1} \delta\Gamma = 2\mathrm{i} \int \mathrm{d}^{2n}x\, \eta(x) G(x). \tag{4.11}$$

Returning to (2.12), one obtains

$$\int [\mathrm{d}\bar{\psi}\, \mathrm{d}\psi] \exp\left(-\int \mathrm{d}^{2n}x\, \bar{\psi}(\mathrm{i}\not{\partial} + \mathrm{i}\not{V})\psi\right)$$
$$= \mathrm{e}^{\Delta\Gamma} \int [\mathrm{d}\bar{\psi}\, \mathrm{d}\psi] \exp\left(-\int \mathrm{d}^{2n}x\, \bar{\psi}(\mathrm{i}\not{\partial} + \mathrm{i}\not{V} - \gamma_5 \not{\partial}\eta(x))\psi\right). \tag{4.12}$$

Taking the derivative with respect to $\eta(x)$ and put $\eta(x) \to 0$, one arrives at the famous ABJ anomaly formula:

$$\langle \partial_\mu J^5_\mu \rangle = \langle \partial_\mu (\bar{\psi}\gamma_\mu \gamma_5 \psi) \rangle = -2\mathrm{i} G(x) \tag{4.13}$$

where $\langle\ \rangle$ denotes the quantum average.

5. A new regularisation formula

We are now in a position to calculate the anomalous effective action induced by chiral transformation shown in (3.12). Since only the zero modes are involved, actually no divergence problem occurs. So the following regulatisation scheme can be proposed.

$$\mathrm{Tr}\, \beta\gamma_5 \vartheta = \{\mathrm{Tr}\, \beta\gamma_5 f(\xi, \mathrm{i}\not{D})\}_{\xi\ \text{independent}} \tag{5.1}$$

$$\mathrm{Tr}\, \beta\gamma_5 \tilde{\vartheta} = \{\mathrm{Tr}\, \beta\gamma_5 f(\xi, (\mathrm{i}\not{D})^{\dagger})\}_{\xi\ \text{independent}} \tag{5.2}$$

where

$$f(\xi, \mathrm{i}\not{D}) = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{e}^{-\xi z}}{z - \mathrm{i}\not{D}} \mathrm{d}z \tag{5.3}$$

is a substitution for the projection operator of zero modes with the contour C on a complex z plane being selected so that it bypasses the origin and $\mathrm{e}^{-\xi z}/(z - \mathrm{i}\not{D})$ is bounded on C, as shown in figure 1.

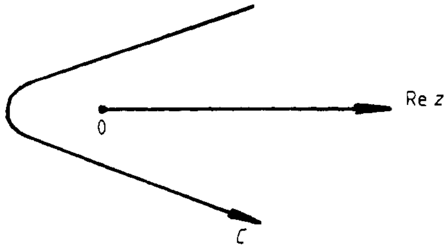

Figure 1.

The implication of the ξ-independent prescription in (5.1) is obviously that it picks only the zero modes of $\mathrm{i}\not{D}$ while neglecting all the non-zero ones. In the following our task is to pick the residue at $z=0$ which is ξ independent. We denote explicitly

$$\mathrm{i}\not{D} = \mathrm{i}\not{D}(a_1, a_2, \ldots, a_{s-1}) = \mathrm{i}\gamma_\mu(\partial_\mu + V_\mu + A_\mu\gamma_5) \tag{5.4}$$

by omitting all the redundant subscripts. Remember $V_\mu^+ = -V_\mu$, $A_\mu^+ = -A_\mu$, so

$$(\mathrm{i}\not{D})^+ = \mathrm{i}\gamma_\mu(\partial_\mu + V_\mu - A_\mu\gamma_5) \tag{5.5}$$

we rewrite

$$\begin{aligned}(z-\mathrm{i}\not{D})^{-1} &= (z+(\mathrm{i}\not{D})^+)(z+(\mathrm{i}\not{D})^+)^{-1}(z-\mathrm{i}\not{D})^{-1}\\ &= (z+(\mathrm{i}\not{D})^+)[z^2 - 2\mathrm{i}z\not{A}\gamma_5 - (\mathrm{i}\not{D})(\mathrm{i}\not{D})^+]^{-1}\end{aligned} \tag{5.6}$$

in the momentum representation

$$(z-\mathrm{i}\not{D})^{-1} = (z-\not{K}+\not{L}^+)(z^2-k^2-Q+2k\not{L}^+)^{-1} \tag{5.7}$$

where

$$\not{L} = \mathrm{i}\not{k} - \mathrm{i}\not{V} + \mathrm{i}\not{A}\gamma_5 \tag{5.8}$$

$$\begin{aligned} Q &= \not{L}\not{L}^+ + 2\mathrm{i}z\not{A}\gamma_5\\ \bar{Q} &= \not{L}^+\not{L} + 2\mathrm{i}z\not{A}\gamma_5\end{aligned} \tag{5.9}$$

with the derivative symbol d only acting on A or V.

Furthermore, an expansion

$$(z^2-k^2-Q+2k\not{L}^+)^{-1} = \sum_{l=0}^{\infty}\frac{1}{(z^2-k^2)^{l+1}}(Q-2k\not{L}^+)^l \tag{5.10}$$

is used without worrying about its convergence property. This is because we confine ourselves in keeping ξ-independent terms and taking the trace at the final stage, so only finite terms in (5.10) survive.

After a long and careful calculation (changing $-\mathrm{i}z \to z$) we arrive at a neat formula

$$\begin{aligned}\mathrm{Tr}(\beta\gamma_5\vartheta) &= \Big(\mathrm{Tr}\,\beta\gamma_5 f(\xi, \mathrm{i}\not{D})\Big)_{\xi\text{ independent}}\\ &= \frac{1}{(4\pi)^n}\frac{1}{n!}\sum_{m=0}^{n} B(m+1, n+1)\\ &\quad\times \mathrm{Tr}\left[\beta(x)\gamma_5\frac{1}{(2m)!}\left(\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}}\sum_{\substack{l_1+l_2\\=m+n}} Q_+^{l_1}Q_-^{l_2}\right)_{z=0}\right]\end{aligned} \tag{5.11}$$

in which ($Q \to -Q_-$, $\bar{Q} \to -Q_+$ by deleting irrelevant terms)

$$Q_\pm = F^{(V)}_{\mu\nu}\sigma_{\mu\nu} \pm F^{(A)}_{\mu\nu}\sigma_{\mu\nu}\gamma_5 + 2z\not{A}\gamma_5 \tag{5.12}$$

$$F^{(V)}_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu] + [A_\mu, A_\nu] \tag{5.13}$$

$$F^{(A)}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, V_\nu] + [V_\mu, A_\nu] \tag{5.14}$$

with $\sigma_{\mu\nu} = \frac{1}{4}[\gamma_\mu, \gamma_\nu]$ and $B(m+1, n+1) = \Gamma(m+1)\Gamma(n+1)/\Gamma(m+n+2)$ is the usual beta function. The symbol $[1/(2m)!]\mathrm{d}^{2m}/\mathrm{d}z^{2m}|_{z=0}$ implies that only the coefficients of z^{2m} terms in $Q_+^{l_1}Q_-^{l_2}$ need to be read out. It is easy to obtain $\mathrm{Tr}\,\beta\gamma_5\tilde{\vartheta}$ in (5.2) by changing the sign of A_μ (see (5.5)), which corresponds to $Q_+ \to Q_-$ in (5.11). Therefore, we have the final result of (3.12)

$$\delta\Gamma(V, A; \beta) = \frac{1}{(4\pi)^m}\frac{1}{n!}\sum_{m=0}^{n} B(m+1, n+1)$$
$$\times \mathrm{Tr}\left[\beta(x)\gamma_5 \frac{1}{(2m)!}\left(\frac{\mathrm{d}^{2m}}{\mathrm{d}z^{2m}} \sum_{\substack{l_1+l_2\\=m+n}} (Q_+^{l_1}Q_-^{l_2} + Q_-^{l_1}Q_+^{l_2})\right)_{z=0}\right] \tag{5.15}$$

All the above calculations are performed in $2n$-dimensional Euclidean space. For the four-dimensional case ($n=2$, $m=0, 1, 2$), it is quite easy to check that

$$\delta\Gamma(V, A; \beta) = -\frac{1}{(4\pi)^2}\mathrm{Tr}\{\beta(x)\varepsilon_{\mu\nu\rho\sigma}[F^{(V)}_{\mu\nu}F^{(V)}_{\rho\sigma} + \tfrac{1}{3}F^{(A)}_{\mu\nu}F^{(A)}_{\rho\sigma} + \tfrac{32}{3}A_\mu A_\nu A_\rho A_\sigma$$
$$- \tfrac{8}{3}(F^{(V)}_{\mu\nu}A_\rho A_\sigma + A_\mu F^{(V)}_{\nu\rho}A_\sigma + A_\mu A_\nu F^{(V)}_{\rho\sigma})]\}. \tag{5.16}$$

It is just the well known non-Abelian chiral anomaly deduced by Bardeen (1969), Alvarez-Gaumé and Ginsparg (1984), Bardeen and Zumino (1984) and Gipson (1986).

For deriving the Abelian anomaly, we simply set $A_\mu = 0$, $\beta(x) = \mathrm{i}\eta(x)t$ (see § 4), the ABJ anomaly term can be read off from (5.11) with $m=0$ only:

$$G(x) = \tilde{G}(x) = \frac{1}{(4\pi)^n}\frac{(-\mathrm{i})^n}{n!}\mathrm{tr}(\varepsilon_{\mu_1\nu_1\ldots\mu_n\nu_n}F_{\mu_1\nu_1}\ldots F_{\mu_n\nu_n}) \tag{5.17}$$

where

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu + [V_\mu, V_\nu]. \tag{5.18}$$

6. Summary and discussion

The ingredients of our treatment for anomaly by path integral approach are composed of four linking parts.

(i) By introducing the comoving representation, one is able to calculate the successive ratio product of determinants for a finite chiral transformation instead of calculating the change in functional measure for an infinitesimal transformation.

(ii) The Dirac operator accompanying the chiral transformation is non-Hermitian in general, but it can be diagonalised in comoving representation with a matrix element being a complex number. It is the argument of this diagonalised element that plays a central role in the anomaly.

(iii) We try to explore the implication of the Wess–Zumino consistency condition by a topological argument that it is only the contributions of zero modes of the Dirac operator that need to be considered in evaluation of the anomaly.

(iv) Correspondly, a new regularisation scheme is proposed which only picks the contribution of zero modes and leads to a unified formula as shown in (5.15). Though it is rather tedious to derive, it is easy to use.

Some discussion is in order.

(a) There is no worry about the non-Hermitian property of the Dirac operator: it can be treated in a similar way as that of a Hermitian one. In particular the final formula (5.15) shows no serious difference in these two cases.

(b) There is no ambiguity in the choice of functional measure or regularisation scheme now. Our regularisation is finite and unique in the sense of only picking the contribution of zero modes. Actually, it seems to us that the Wess-Zumino condition prevailing in the literature ensures correctly counting the zero modes while suppressing the non-zero ones. Moreover, in our point of view, the well known regularisation scheme proposed by Fujikawa for Abelian anomaly with Hermitian ($i\not{D}$):

$$G(x) = \lim_{M\to\infty}\left(\sum_n \phi_n^{\dagger}(x)\gamma_5 \exp[-(i\not{D})^2/M^2]\phi_n(x)\right) \tag{6.1}$$

is essentially the following:

$$G(x) = \left(\sum_n \phi_n^{\dagger}(x)\gamma_5 \exp[-(i\not{D})^2/M^2]\phi_n(x)\right)_{M\ \text{independent}} \tag{6.2}$$

which stresses the selection of zero modes in similar style as that in this paper.

(c) Last, but not least, we prefer to use the symbol of equality '=' rather than transformation '→' thoughout the formulation. It is advantageous to obtain the anomaly formula (4.13) with the symbol ⟨ ⟩, to treat some special problems such as the Schwinger model or (Abelian or non-Abelian) bosonisation in two dimensions (see appendix 3), or to derive the Wess-Zumino term in the general case. The latter problem will be discussed in a separate paper where the relation between the path integral method and a differential geometry approach to the anomaly will be further clarified.

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Appendix 1. How to avoid the 0/0 ambiguity in calculation

As pointed out after (2.37), the zero diagonal elements of $i\not{D}$ will cause difficulty in the calculation. To avoid this, we introduce a small chiral symmetry breaking term from the very beginning, i.e. redefine $i\not{D}_\varepsilon = i\not{D} + \varepsilon\gamma_5$. Then, for example, in the case of the Abelian anomaly, we have $i\not{D}'_\varepsilon = \exp(-i\beta(x)t\gamma_5)i\not{D}_\varepsilon \exp[-i\beta(x)t\gamma_5]$ and denote $\{|\phi'_n(\varepsilon)\rangle\}$ and $\{|\tilde{\phi}'_n(\varepsilon)\rangle\}$ as the complete set of eigenvectors of $\Delta'_\varepsilon = (i\not{D}'_\varepsilon)^{\dagger}(i\not{D}'_\varepsilon)$ and $\tilde{\Delta}'_\varepsilon = (i\not{D}'_\varepsilon)(i\not{D}'_\varepsilon)^{\dagger}$ respectively. Corresponding to (2.21) and (2.22), one obtains

$$i\not{D}'_\varepsilon|\phi'_n(\varepsilon)\rangle = \lambda_n(\varepsilon)\exp(i\theta_n(t,\varepsilon))|\tilde{\phi}'_n(\varepsilon)\rangle \tag{A1.1}$$

$$(i\not{D}'_\varepsilon)^{\dagger}|\tilde{\phi}'_n(\varepsilon)\rangle = \lambda_n(\varepsilon)\exp(-i\theta_n(t,\varepsilon))|\phi'_n(\varepsilon)\rangle \tag{A1.2}$$

and

$$\langle\tilde{\phi}'_m(\varepsilon)|i\not{D}'_\varepsilon|\phi'_n(\varepsilon)\rangle = \lambda_n(\varepsilon)\exp(i\theta_n(t,\varepsilon))\delta_{m,n} \tag{A1.3}$$

where

$$\lambda_n^2(\varepsilon) = \lambda_n^2 + \varepsilon^2. \tag{A1.4}$$

Now, even the original zero diagonal elements of $\mathrm{i}\not{D}'$ become $\varepsilon \exp(\mathrm{i}\theta_i(t, \varepsilon))$ or $-\varepsilon \exp(\mathrm{i}\theta_i(t, \varepsilon))$ in the $\mathrm{i}\not{D}'_\varepsilon$ representation. Everything is well defined. Then at the final stage, let $\varepsilon \to 0$ and all formulae in the text remain valid.

Appendix 2. The index as a winding number

As discussed in § 3, the Wess–Zumino condition implies that a continuous transformation along a closed path in the group manifold will map into a closed path on the complex plane for each diagonal element of the Dirac operator. Depending on this property, we manage to introduce a winding number which bears a direct relationship to the index of this Dirac operator.

Denoting the operator after s steps as

$$\mathrm{i}\not{D}_s = \mathrm{i}\not{D}(a_1, \ldots, a_s) = \mathrm{i}(\not{\partial} + \not{V}_s + \gamma_5 \not{A}_s) \tag{A2.1}$$

and looking at a finite chiral transformation $e^{v\gamma_5}$ starting from this point and going back to it, we have

$$\exp(v(\varphi)\gamma_5) = \exp(v\hat{P}_+ - v\hat{P}_-) = e^v\hat{P}_+ + e^{-v}\hat{P}_- \equiv g\hat{P}_+ + g^{-1}\hat{P}_- \tag{A2.2}$$

with

$$\hat{P}_\pm \equiv \tfrac{1}{2}(1 \pm \gamma_5) \tag{A2.3}$$

and a parameter φ chosen so that $g(\varphi = 0) = g(\varphi = 2\pi) = \mathbb{1}$, $e^v = g(\varphi, x) \in G$, and

$$\mathrm{d}v = g^{-1}\,\mathrm{d}_\varphi g \qquad -\mathrm{d}v = g\,\mathrm{d}_\varphi g^{-1} \tag{A2.4}$$

$$-\mathrm{d}v\,\gamma_5 = -\mathrm{d}v\,\hat{P}_+ + \mathrm{d}v\,\hat{P}_- = g\,\mathrm{d}_\varphi g^{-1}\,\hat{P}_+ + g^{-1}\,\mathrm{d}_\varphi g\,\hat{P}_-. \tag{A2.5}$$

Denoting

$$\exp(-v(\varphi)\gamma_5) = g^{-1}\hat{P}_+ + g\hat{P}_- \tag{A2.6}$$

$$\mathrm{i}\not{D}_s^\varphi = \exp(-v\gamma_5)\mathrm{i}\not{D}_s \exp(-v\gamma_5) \tag{A2.7}$$

(see (2.16)) and

$$\Delta^\varphi = (\mathrm{i}\not{D}_s^\varphi)^\dagger(\mathrm{i}\not{D}_s^\varphi) \tag{A2.8}$$

$$\tilde{\Delta}^\varphi = (\mathrm{i}\not{D}_s^\varphi)(\mathrm{i}\not{D}^\varphi)^\dagger \tag{A2.9}$$

we have their eigenfunctions written as

$$\phi_{sn}^\varphi = \exp(\tfrac{1}{2}\mathrm{i}\theta_{sn}(\varphi)) \exp(v(\varphi)\gamma_5)\phi_{sn} \tag{A2.10}$$

$$\tilde{\phi}_{sn}^\varphi = \exp(-\tfrac{1}{2}\mathrm{i}\theta_{sn}(\varphi)) \exp(-v(\varphi)\gamma_5)\tilde{\phi}_{sn}. \tag{A2.11}$$

Here

$$\phi_{sn}^{\varphi=0} = \phi_n(x; a_1, \ldots, a_s) = \phi_{sn} \tag{A2.12}$$

and

$$\tilde{\phi}_{sn}^{\varphi=0} = \tilde{\phi}_n(x; a_1, \ldots, a_s) = \tilde{\phi}_{sn} \tag{A2.13}$$

as shown in (2.25) and (2.26). Also, as in (2.37), one can calculate the argument change $\delta\theta_{sn}$ when $\varphi \to \varphi + d\varphi$ as

$$i\delta\theta_{sn} = -\int d^{2n}x\, \phi^{\varphi}_{sn}\, dv\, \gamma_5 \varphi^{\varphi}_{sn} - \int d^{2n}x\, \tilde{\phi}^{\varphi\dagger}_{sn}\, dv\, \gamma_5 \phi^{\varphi}_{sn}. \tag{A2.14}$$

Substituting $-dv\gamma_5$ from (A2.5) and (A2.10), (A2.11) into (A2.14), one obtains

$$i\delta\theta_{sn} = -\int d^{2n}x(\phi^{(+)\dagger}_{sn} g^{-1}\, d_\varphi g\, \phi^{(+)}_{sn} - \phi^{(-)\dagger}_{sn} g^{-1}\, d_\varphi g\, \phi^{(-)}_{sn} + \tilde{\phi}^{(+)\dagger}_{sn} g^{-1}\, d_\varphi g\, \tilde{\phi}^{(+)}_{sn} - \tilde{\phi}^{(-)\dagger}_{sn} g^{-1}\, d_\varphi g\, \tilde{\phi}^{(-)}_{sn}) \tag{A2.15}$$

where $\phi^{(\pm)}_{sn} = \hat{P}_\pm \phi_{sn}$, etc, and the relation (A2.4) have been used.

Now we see that the integration with respect to φ can be performed in (A2.15):

$$\int_{\varphi=0}^{\varphi=2\pi} g^{-1}\, d_\varphi g = 2\pi i \tag{A2.16}$$

so

$$i\Delta\theta_{sn} = i\oint \delta\theta_{sn} n(\varphi) = -2\pi i \int d^{2n}x[(\phi^{(+)\dagger}_{sn}\phi^{(+)}_{sn} - \phi^{(-)\dagger}_{sn}\phi^{(-)}_{sn}) + (\tilde{\phi}^{(+)\dagger}_{sn}\tilde{\phi}^{(+)}_{sn} - \hat{\phi}^{(+)\dagger}_{sn}\tilde{\phi}^{(-)}_{sn})]. \tag{A2.17}$$

If $n > 0$, when there is a $\phi^{(+)}_{sn}$, there must be a $\phi^{(-)}_{sn}$ according to the mapping relation, so only the contributions of $n = 0$ modes survive. In these cases, $\phi^{(\pm)}_{sn}$ and $\tilde{\phi}^{(\pm)}_{sn}$ should be understood as the zero modes of $i\not{D}^{(\pm)}_s \equiv i\not{D}_s \frac{1}{2}(1 \pm \gamma_5)$ and $(i\not{D}_s)^{\dagger(\pm)} = (i\not{D}_s)^\dagger \frac{1}{2}(1 \pm \gamma_5)$ respectively.

Hence we have the change of effective action during the rotation of $\phi: 0 \to 2\pi$:

$$\begin{aligned} \Delta\Gamma_s &= -i\Delta\theta_{s0} = +2\pi i(n_+ - n_- + \tilde{n}_+ - \tilde{n}_-) \\ &\equiv 2\pi i[\text{ind}(i\not{D}_s) + \text{ind}(i\not{D}_s)^\dagger] \end{aligned} \tag{A2.18}$$

where

$$\text{ind}(i\not{D}_s) = n_+ - n_- \qquad \text{ind}(i\not{D}_s)^\dagger = \tilde{n}_+ - \tilde{n}_- \tag{A2.19}$$

is the index of operator $i\not{D}_s$ or $(i\not{D}_s)^\dagger$ defined as the difference of the numbers of zero modes with opposite chirality. Thus we see that the index of $i\not{D}_s$ or $(i\not{D}_s)^\dagger$ could be identified to a winding number which characterises the mapping of the closed path in a group manifold into the complex plane for diagonal elements of the Dirac operator. Only the contribution of the argument for zero modes is non-trivial. This topological explanation allows one to interpret the meaning of the Wess-Zumino condition as a constraint of picking only the contributions of zero modes in calculating the anomaly.

Appendix 3. The Schwinger model

The Schwinger model is defined as the QED in $(1+1)$-dimensional space. Let us also turn to Euclidean space:

$$\mathcal{L} = \bar{\psi} i\not{D}\psi \tag{A3.1}$$

where

$$i\not{D} = i(\not{\partial} - ie\not{A}) \tag{A3.2}$$

with $\gamma_1 = \sigma_1$, $\gamma_2 = \sigma_2$, $\gamma_5 = i\gamma_1\gamma_2 = -\sigma_3$. Notice that now we have an important relation:

$$\gamma_\mu \gamma_5 = i\varepsilon_{\mu\nu}\gamma_\nu \tag{A3.3}$$

so that it is possible to decouple the fermion field from the gauge field by chiral transformation in the generating functional. Actually, according to (4.5),

$$i\not{D}' = \exp(-i\eta(x)t\gamma_5) i\not{D} \exp(-i\eta(x)t\gamma_5) = i(\not{\partial} - ie\not{A}) + t\not{\partial}\eta(x)\gamma_5. \tag{A3.4}$$

If $\eta(x)$ satisfies the following decoupling condition by using (A3.3):

$$e\gamma_\mu A_\mu = i\varepsilon_{\mu\nu}\gamma_\mu \partial_\nu \eta(x) \tag{A3.5}$$

then

$$i\not{D}' = i\not{\partial} + e(1-t)\not{A}. \tag{A3.6}$$

At $t=1$,

$$i\not{D}^{t=1} = i\not{\partial}. \tag{A3.7}$$

Notice, however, (A3.5) implies $\eta(x)$ being imaginary. For using the formula (5.17), we put V_μ in (5.18) as

$$V_\mu(t) = -ie(1-t)A_\mu \tag{A3.8}$$

then (5.17) gives

$$G(x,t) = \tilde{G}(x,t) = \frac{-e}{4\pi}(1-t)\varepsilon_{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu). \tag{A3.9}$$

Substituting it into (4.11), one obtains

$$\begin{aligned} \Delta\Gamma &= 2i \int_0^1 dt \int d^2x\, \eta(x) G(x,t) \\ &= -i\frac{e}{2\pi}\int d^2x\, \partial_\nu\eta(x)\varepsilon_{\mu\nu}A_\mu = -\frac{e^2}{2\pi}\int d^2x\, A_\mu A_\mu \end{aligned} \tag{A3.10}$$

so (4.12) implies a vector boson with mass $e/\sqrt{\pi}$ emerges:

$$\begin{aligned} &\int [d\bar{\psi}\, d\psi] \exp\left(-\int d^2x\, \bar{\psi}(i\not{D} + e\not{A})\psi\right) \\ &\quad = \left[\exp\left(-\frac{e^2}{2\pi}\int d^2x\, A_\mu A_\mu\right)\right]\left[\int [d\bar{\psi}\, d\psi]\exp\left(-\int d^2x\, \bar{\psi} i\not{\partial}\psi\right)\right] \end{aligned} \tag{A3.11}$$

which was first derived by Roskies and Schaponsnik (1981) by the path integral method.

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